

RATIONAL APPROXIMATION SCHEMES FOR SOLUTIONS OF THE FIRST AND SECOND ORDER CAUCHY PROBLEM

PATRICIO JARA

(Communicated by Nigel J. Kalton)

ABSTRACT. The purpose of this paper is to give sharp error estimates for regularized versions of \mathcal{A} -stable rational approximations of C -regularized semigroups such as the Backward Euler and Crank-Nicolson scheme among others. The main tools used are those developed by P. Brenner and V. Thomée for strongly continuous semigroups together with a regularized version of the Hille-Phillips functional calculus.

INTRODUCTION

Over the years there have been different approaches to study the abstract Cauchy problem (ACP) $u'(t) = Au(t)$, $u(0) = x$, where A is a linear operator on a Banach space X . The most powerful approach is through the theory of strongly continuous semigroups, see [9, 10, 23]. Other examples are *bi-continuous semigroups* [19], *distribution semigroups* [20], *integrated semigroups* [1], *convolution semigroups* [5], and *C -regularized semigroups* [6, 7, 8]. However, aside from the strongly continuous case, these semigroup theories have few qualitative results dealing with the problem of approximating solutions of (ACP) by means of time and space discretization. In the literature one finds that the work initiated by R. Hersh and T. Kato [12] and continued by P. Brenner and V. Thomée [3] provide the best general time discretization results with error estimates for rational approximations of strongly continuous semigroups. In [16], it is shown that these results can be fully extended for bi-continuous semigroups. Therefore, it is natural to ask if similar results can be obtained for other generalizations of strongly continuous semigroups.

In particular, one can look at C -regularized semigroups or C -semigroups. Exponentially bounded C -semigroups were introduced by G. Da Prato in [6], and by E. B. Davies and M. M. H. Pang in [7]; the general case was further developed by R. deLaubenfels among others. References and a comprehensive introduction to C -semigroups can be found in [8]. The C -semigroup approach provides a powerful tool to study (ACP) for operators not generating strongly continuous semigroups, including generators of bi-continuous, integrated and distributional semigroups, see [19, 20]. Moreover, it allows the unified study of well-posed and ill-posed problems.

2000 *Mathematics Subject Classification*. Primary 65M12, 65M15; Secondary 47D60, 44A45, 47D62.

Key words and phrases. C -regularized semigroups, Hille-Phillips functional calculus, time discretization, Pade approximants, Backward Euler, Crank-Nicolson, RadauIIA.

This paper shows that the framework of C -semigroups allows one to extend the classical theory for time discretization schemes, defined via rational approximations to the exponential, from the strongly continuous case to the regularized case. Thus, one can approximate solutions of abstract Cauchy problems when the operator A generates a C -semigroup of type (M, ω) , by using \mathcal{A} -stable rational functions that approximate the exponential, with error estimates for smooth initial data.

1. BANACH ALGEBRAS OF NORMALIZED FUNCTIONS OF BOUNDED VARIATION

Often, the Hille-Phillips functional calculus is given in terms of measures, see [11, 14]. In order to be able to use integration by parts, we prefer the original approach of R. S. Phillips via functions of bounded variation, see [24]. This section shows precisely in what sense those approaches are equivalent. Moreover, it also shows an adaptation of a method developed by P. Brenner and V. Thomée in order to obtain norm estimates of such functions.

The space of normalized functions of bounded variation is denoted by NBV_{loc} for which $\text{Var}_I(\alpha)$ is the total variation of α on an interval I . Notice that $V_\alpha(t) := \text{Var}_{[0,t]}(\alpha) \in \text{NBV}_{\text{loc}}$ for $\alpha \in \text{NBV}_{\text{loc}}$. The Riemann-Stieltjes convolution on NBV_{loc} is defined by $(\alpha * \beta)(t) := \int_0^t \alpha(t-s)d\beta(s)$. It can be shown that $V_{\alpha*\beta}(t) \leq V_\alpha(t)V_\beta(t)$, where t is not a point of discontinuity for α nor β . For a proof see [26].

Let

$$(1.1) \quad C_0(\mathbb{R}_0^+, \omega) := \left\{ x : \mathbb{R}_0^+ \rightarrow \mathbb{C} \text{ continuous and } \lim_{t \rightarrow \infty} e^{\omega t} x(t) = 0 \right\}$$

with $\|x\|_{\infty, \omega} := \sup_{t \in \mathbb{R}_0^+} |e^{\omega t} x(t)|$, $\omega \in \mathbb{R}$. Since the map $\varphi_\omega : C_0(\mathbb{R}_0^+, \omega) \rightarrow C_0(\mathbb{R}_0^+)$ defined by $\varphi_\omega(f)(t) := f(t)e^{\omega t}$ is an isometric isomorphism, a straightforward calculation yields that the map $\varphi_\omega^* : \text{NBV}_{\text{loc}} \rightarrow \text{NBV}^\omega$ defined by $\varphi_\omega^*(\alpha)(t) = \int_0^t e^{-\omega s} d\alpha(s)$ is an isometric isomorphism as well, where

$$(1.2) \quad \text{NBV}^\omega := \left\{ \alpha \in \text{NBV}_{\text{loc}} : \int_0^\infty e^{\omega t} dV_\alpha(t) < \infty \right\} \text{ and } \|\alpha\|_\omega = \int_0^\infty e^{\omega t} dV_\alpha(t).$$

In this way, the following representation is obtained.

Theorem 1.1 (Riesz). *Let $\omega \in \mathbb{R}$. If $\Psi \in (C_0(\mathbb{R}_0^+, \omega))'$ then there exist a unique $\alpha_\omega \in \text{NBV}^\omega$ such that $\langle f, \Psi \rangle = \int_0^\infty f(t) d\alpha_\omega(t)$, for every $f \in C_0(\mathbb{R}_0^+, \omega)$.*

It follows that, for all $\omega \in \mathbb{R}$, the space NBV^ω is a Banach algebra with the Stieltjes-convolution as product and $\|\alpha\|_\omega := \int_0^\infty e^{\omega t} dV_\alpha(t)$. Furthermore, if

$$(1.3) \quad F_\omega := \left\{ f_\alpha : f_\alpha(z) = \int_0^\infty e^{zt} d\alpha(t) \text{ if } \text{Re}(z) \leq \omega, \alpha \in \text{NBV}^\omega \right\},$$

then the operator $\Phi : \text{NBV}^\omega \rightarrow F_\omega$ defined by $\Phi(\alpha) = f_\alpha$ is an algebra isomorphism. Moreover, if $\|f_\alpha\| := \|\alpha\|_\omega$ then F_ω is a Banach algebra and the inclusion $F_\omega \subset F_\kappa$ holds for $\omega \geq \kappa$. Notice that f_α is the Laplace-Stieltjes transform of $\alpha \in \text{NBV}^\omega$; and since for a given function f , the variation norm of the inverse Laplace-Stieltjes transform α is difficult to estimate directly, we adapt a method from [3] which allows to obtain sharp estimates for $\|\alpha\|_\omega$ in terms of f by considering the Fourier-Stieltjes transform. The Fourier transform of $f \in L^1(\mathbb{R})$ is defined by $\hat{f}(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-its} f(s) ds$, and the Fourier-Stieltjes transform of $\alpha \in \text{NBV}(\mathbb{R})$ is defined by $\mathcal{F}[\alpha](t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-its} d\alpha(s)$ ($t \in \mathbb{R}$). Notice that if there is $f \in L^1(\mathbb{R})$ such

that $\alpha(t) = \int_0^t f(s)ds + \alpha(0)$, then $\mathcal{F}[\alpha] = \widehat{f}$ and the Riemann-Lebesgue Theorem implies that $\mathcal{F}[\alpha] \in C_0(\mathbb{R})$. Moreover, in this case, $\text{Var}(\alpha) = \|f\|_{L^1(\mathbb{R})}$. Also, NBV_{loc} can be continuously embedded in $\text{NBV}(\mathbb{R})$ by setting $\alpha \equiv 0$ on \mathbb{R}^- .

Lemma 1.2. *Let $\alpha \in \text{NBV}_{\text{loc}}$ and $f_0(t) := \mathcal{F}[\alpha](-t) \in L^2(\mathbb{R})$. Then*

$$\alpha(t) = \int_0^t \widehat{f}_0(w)dw.$$

If in addition $f'_0 \in L^2(\mathbb{R})$, then $\widehat{f}_0 \in L^1(\mathbb{R}_0^+)$ and

$$(1.4) \quad \text{Var}(\alpha) = \|\widehat{f}_0\|_{L^1(\mathbb{R}_0^+)} \leq \sqrt{\pi} \|f_0\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|f'_0\|_{L^2(\mathbb{R})}^{\frac{1}{2}}.$$

Proof. Since NBV_{loc} can be continuously embedded into $\text{NBV}(\mathbb{R})$ by setting $\alpha \equiv 0$ on \mathbb{R}^- , the inversion formula for the Fourier-Stieltjes transform asserts that $\alpha(t) - \alpha(0) = \frac{1}{\sqrt{2\pi}} \lim_{R \rightarrow \infty} \int_{-R}^R \mathcal{F}[\alpha](s) \frac{e^{ist} - 1}{is} ds$. It follows by Fubini's Theorem that

$$(1.5) \quad \sqrt{2\pi}\alpha(t) = \lim_{R \rightarrow \infty} \int_{-R}^R \mathcal{F}[\alpha](s) \frac{e^{ist} - 1}{is} ds = \lim_{R \rightarrow \infty} \int_0^t \int_{-R}^R f_0(s) e^{-iws} ds dw.$$

On the other hand, since $\text{l.i.m.}_{R \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^R f_0(s) e^{-iws} ds = \widehat{f}_0(w)$ (where l.i.m. denotes the limit in the $L^2(\mathbb{R})$ sense), and strong convergence implies weak convergence in $L^2(\mathbb{R})$, one obtains that $\sqrt{2\pi} \langle \widehat{f}_0, \chi_{[0,t]} \rangle = \lim_{R \rightarrow \infty} \int_0^t \int_{-R}^R f_0(s) e^{-iws} ds dw$. It follows from (1.5) that $\alpha(t) = \int_0^t \widehat{f}_0(w)dw$. Notice that $\widehat{f}_0 = 0$ a.e. on \mathbb{R}^- since $\alpha(t) = 0$ for all $t \leq 0$. Now, let $f'_0 \in L^2(\mathbb{R})$. Carlson's inequality (see [4]) states that if $g \in L^2(\mathbb{R}_0^+)$, and $h(s) := sg(s) \in L^2(\mathbb{R}_0^+)$, then $\|g\|_{L^1(\mathbb{R}_0^+)} \leq \sqrt{\pi} \|g\|_{L^2(\mathbb{R}_0^+)}^{\frac{1}{2}} \|h\|_{L^2(\mathbb{R}_0^+)}^{\frac{1}{2}}$. From $f_0 \in L^2(\mathbb{R})$, it follows that $\widehat{f}'_0(s) = is\widehat{f}_0(s)$. Thus \widehat{f}'_0 and \widehat{f}_0 are in $L^2(\mathbb{R}_0^+)$. Furthermore, Parseval's theorem together with Carlson's inequality imply that $\|\widehat{f}_0\|_{L^1(\mathbb{R}_0^+)} \leq \sqrt{\pi} \|f_0\|_{L^2(\mathbb{R}_0^+)}^{\frac{1}{2}} \|f'_0\|_{L^2(\mathbb{R}_0^+)}^{\frac{1}{2}}$. Therefore $\widehat{f}_0 \in L^1(\mathbb{R}^+)$. Since $\alpha(t) = \int_0^t \widehat{f}_0(w)dw$, it follows that $\text{Var}(\alpha) = \|\widehat{f}_0\|_{L^1}$. \square

Theorem 1.3. *Let $\alpha \in \text{NBV}_{\text{loc}}$ and $q \in \mathbb{N}$. Suppose that for each $1 \leq k \leq q$ there exist $\beta_k \in \text{NBV}_{\text{loc}}$ such that $f_k(t) = \frac{g(t)}{t^k}$, where $g(t) := \mathcal{F}[\alpha](-t)$, and $f_k(t) := \mathcal{F}[\beta_k](-t)$. If f_k satisfies the conditions of Lemma 1.2 for $1 \leq k \leq q$, then*

$$(1.6) \quad \text{Var}(\beta_k) = \|I^{k-1}[\alpha]\|_{L^1(\mathbb{R}^+)},$$

where I^k denotes k^{th} -antiderivative; i.e. $I[\alpha](t) := \int_0^t \alpha(\xi)d\xi$, and $I^0 := \text{id}$ on NBV_{loc} . Moreover, for $1 \leq k \leq q$,

$$(1.7) \quad \lim_{t \rightarrow \infty} I^{k-1}[\alpha](t) = 0.$$

Proof. As before, $\beta_k \equiv 0$ on \mathbb{R}^- . It follows from Lemma 1.2 that $\widehat{f}_k \in L^1(\mathbb{R}^+)$ for $1 \leq k \leq q$. It follows that $\widehat{f}_k^{(m)}(t) = [(-i(\cdot))^m \widehat{f}_k(\cdot)](t) = (-i)^m [(\cdot)^m \widehat{f}_k(\cdot)](t) = (-i)^m \widehat{f}_{k-m}(t)$ for $0 \leq m \leq k$. By Riemann-Lebesgue, $\lim_{t \rightarrow \pm\infty} \widehat{f}_k^{(m)}(t) = 0$ for $0 \leq m \leq k$. Since $(-it)^k f_k(t) = (-i)^k g(t)$, it follows that $\widehat{f}_k^{(k)}(t) = (-i)^k \widehat{g}(t)$. Thus $\widehat{f}_k^{(k-1)}(t) = (-i)^k \int_{-\infty}^t \widehat{g}(s)ds$. Now, from the proof of Lemma 1.2, $\int_{-\infty}^t \widehat{g}(s)ds = 0$ for all $t \leq 0$. In this way, by integrating k -times, $\widehat{f}_k(t) = (-i)^k I^k[\widehat{g}](t) =$

$(-i)^k I^{k-1}[\alpha](t)$. Thus, (1.6) follows from the fact that $\text{Var}(\beta_k) = \|\widehat{f}_k\|_{L^1(\mathbb{R}_0^+)}$ by Lemma 1.2, and (1.7) follows by applying the Riemann-Lebesgue Theorem. \square

We conclude this section by showing that the results above provide a useful tool in order to estimate the variation norm of the inverse Laplace-Stieltjes transform of \mathcal{A} -stable rational functions. Recall that a rational function r is called \mathcal{A} -stable if $|r(z)| \leq 1$ for $\text{Re}(z) \leq 0$; and if $r(z) = e^z + o(z)$ as $z \rightarrow 0$, then r is said to be \mathcal{A} -acceptable. Moreover, r is an approximation to the exponential of order $q \geq 1$ if $r(z) = e^z + O(|z|^{q+1})$ as $z \rightarrow 0$. Consider the Heaviside normalized functions of bounded variation defined by $H_0(s) = 1$ for $s > 0$, and $H_0(0) = 0$; and $H_t(s) = 1$ for $s > t$, $H_t(t) = \frac{1}{2}$, and $H_t(s) = 0$ if $s \in [0, t)$. It follows that the functions

$$(1.8) \quad z \rightarrow e^{tz} = \int_0^\infty e^{sz} dH_t(s) \in F_\omega \quad (\omega \in \mathbb{R}, t \geq 0).$$

In particular, the constant functions are in F_ω for all $\omega \in \mathbb{R}$. Furthermore, rational functions r that are bounded for $\text{Re}(z) \leq \omega$ are in F_ω . Moreover, if $t > 0$ and $n \in \mathbb{N}$, then $r^n\left(\frac{t}{n}z\right) = \int_0^\infty e^{zs} d\alpha_{n,t}(s)$, where $\alpha_{n,t}(s) := \alpha^{n*}\left(\frac{n}{t}s\right)$ and α^{n*} denotes the n -th convolution product $\alpha * \dots * \alpha$ for $\alpha \in \text{NBV}^\omega \subset \text{NBV}_{\text{loc}}$. Finally, if r is an \mathcal{A} -stable rational function that approximates the exponential to order $q \geq 1$, then

$$(1.9) \quad f : z \rightarrow \frac{r^n\left(\frac{t}{n}z\right) - e^{zt}}{z^{k+1}} \in F_0$$

for $0 \leq k \leq q$, $n \in \mathbb{N}$ and $t \geq 0$.

Definition 1.4. An \mathcal{A} -acceptable rational function r is said to satisfy the condition (\star) if the following conditions hold:

- (a) $|r(i\xi)| < 1$ for $0 \neq \xi \in \mathbb{R}$ and $|r(\infty)| < 1$;
- (b) There exist positive integers \bar{p}, \bar{q} , where \bar{p} is even, $\bar{p} \geq \bar{q} + 1$, and a positive number γ such that $r(i\xi) = e^{i\xi + \psi(\xi)}$ with $\psi(\xi) = O(|\xi|^{\bar{q}+1})$ as $\xi \rightarrow 0$; and $\text{Re}(\psi(\xi)) \leq -\gamma|\xi|^{\bar{p}}$ for $|\xi| \leq 1$.

It is shown in [25] that if (a) holds and $r(z) = e^z + O(|z|^{q+1})$ as $z \rightarrow 0$, then (b) holds for $\bar{q} = q$ and some $\bar{p} \geq q + 1$.

Examples of \mathcal{A} -stable rational approximations to the exponential of order $q \geq 1$ are Padé approximants such as the Backward Euler, Crank-Nicolson, and RadauIIA scheme; restricted Padé approximants, and the composite exponential approximations developed by A. Iserles in [15], see [3, 16, 25]. An important case is the one provided by the subdiagonal Padé approximants given by $r_j(z) = \frac{v_j(z)}{w_{j+1}(z)}$ since r_j is an \mathcal{A} -stable rational approximation to the exponential of order $q = 2j + 1$ for every $j \in \mathbb{N}_0$, where

$$(1.10) \quad v_j(z) := \sum_{k=0}^j \frac{(2j-k)!}{k!(j-k)!} z^j \quad \text{and} \quad w_{j+1}(z) := \sum_{k=0}^{j+1} \frac{(2j+1-k)!}{k!(j+1-k)!} (-z)^k.$$

In this case, r_0 is the Backward Euler approximation and r_2 is the RadauIIA scheme which are of order $q = 1$ and $q = 5$ respectively. Furthermore, the subdiagonal Padé approximants r_j satisfy the condition (\star) with $\bar{q} = 2j + 1$ and $\bar{p} = 2j + 2$; for details see [25].

Theorem 1.5 (Brenner-Thomé). *If $\alpha \in \text{NBV}^0$ is such that $r(z) := \int_0^\infty e^{sz} d\alpha(s)$ ($\text{Re}(z) \leq 0$) is an \mathcal{A} -stable rational function, then there exists $K > 0$ such that*

$$(1.11) \quad \|\alpha^{n*}\|_0 \leq K\sqrt{n} \quad \text{for all } n \in \mathbb{N}.$$

If, in addition, r is \mathcal{A} -acceptable and satisfies the condition (\star) then there is a constant K such that

$$(1.12) \quad \|\alpha^{n*}\|_0 \leq Kn^{\frac{1}{2}(1-\frac{q+1}{p})} \quad \text{for all } n \in \mathbb{N}.$$

If f is as in (1.9), r approximates the exponential to order $q \geq 1$, and $\theta_q(k) := k\frac{q}{q+1} + \min\left\{0, \frac{k}{q+1} - \frac{1}{2}\right\}$, then there exist $\beta_{k,n,t} \in \text{NBV}_{\text{loc}}$ and $K > 0$ such that $f(z) = \int_0^\infty e^{zs} d\beta_{k,t,n}(s)$ for $\text{Re}(z) \leq 0$ and

$$(1.13) \quad \text{Var}(\beta_{k,n,t}) = \|I^{k-1}[\alpha_{n,t} - H_t]\|_{L^1(\mathbb{R}^+)} \leq K \frac{t^k}{n^{\theta_q(k)}} \quad \text{for all } n \in \mathbb{N},$$

and $1 \leq k \leq q+1$, except for $k = \frac{q-1}{2}$, in which case an additional factor $\ln(n+1)$ should be added to (1.13). Furthermore, if r satisfies (\star) then θ_q can be replaced by $\theta_q^(k) = \frac{k\bar{q}}{\bar{q}+1} + \min\left\{0, (k - \frac{1}{2}(\bar{q}+1))(\frac{1}{\bar{q}+1} - \frac{1}{p})\right\}$.*

Proof. Let $r(z) = \int_0^\infty e^{zs} d\alpha(s)$ ($\text{Re}(z) \leq 0$) for some $\alpha \in \text{NBV}_{\text{loc}}$ and let μ be the bounded regular complex Borel measure associated with the extension of α defined by $\alpha \equiv 0$ on \mathbb{R}^- . Define $r_{(0)}(t) := \tilde{\mu}(t)$ (the Fourier transform of μ) and $m(\tilde{\mu}) := \int_{-\infty}^\infty d|\mu|(t)$. Then, $m(r_{(0)}) = m(\tilde{\mu}) = \|\alpha\|_0$. Thus, $m(r_{(0)}^n) = \|\alpha^{n*}\|_0$, and (1.11) together with (1.12) follow from the proof of Theorem 1 and 2 of [3] with $\omega = 0$. Let $\beta_{k,n,t} \in \text{NBV}_{\text{loc}}$ be such that $f(z) = \int_0^\infty e^{zs} d\beta_{k,n,t}(s)$ ($\text{Re}(z) \leq 0$) from (1.9), and let $\mu_{k,n,t}$ be the bounded regular complex Borel measure associated with $\beta_{k,n,t}$ and let $f_{(0)} = \tilde{\mu}_{k,n,t}$ (the Fourier transform of $\mu_{k,n,t}$). Then, it follows from [3] that $m(f_{(0)}) = m(\tilde{\mu}_{k,n,t}) = \|\beta_{k,n,t}\|_0$. Therefore, (1.13) follows from the proof of Theorem 4 and Remark 3 of [3] with $\omega = 0$. Finally, Theorem 1.3 shows that $\text{Var}(\beta_{k,n,t}) = \|I^{k-1}[\alpha_{n,t} - H_t]\|_{L^1(\mathbb{R}_0^+)}$. \square

Remark 1.6. In [18], M. Kovács shows that similar results can be obtained by using the Laplace-Stieltjes transform instead of the Fourier-Stieltjes transform. Theorem 1.5 shows that Theorem 3.2 of [18] can be sharpened when considering \mathcal{A} -stable rational functions satisfying condition (\star) like the subdiagonal Padé approximants.

2. THE REGULARIZED HILLE-PHILLIPS FUNCTIONAL CALCULUS

In [14, 24], E. Hille and R. Phillips developed a functional calculus for generators A of strongly continuous semigroups T of type (M, ω) on a Banach space X . The functional calculus applies to analytic functions f with a Laplace-Stieltjes representation $f(z) = \int_0^\infty e^{zt} d\alpha(t)$ for $\text{Re}(z) \leq \omega$, where $\alpha \in F_\omega$. In this way, $f(A) := x \rightarrow \int_0^\infty T(t)x d\alpha(t)$ defines a bounded linear operator on X , and the map $\Phi : f \rightarrow f(A)$ defines an algebra homomorphism from F_ω into $\mathcal{L}(X)$, where $\mathcal{L}(X)$ denotes the Banach space of bounded linear operators on X .

This section shows an extension of the Hille-Phillips Functional Calculus to generators of C -regularized semigroups. From now on, C will be a bounded and injective operator defined on X . We follow [8] for the first definitions and basic results on this theory.

A strongly continuous map $W : [0, \infty) \rightarrow \mathcal{L}(X)$ is called a C -regularized semigroup or C -semigroup if the following two conditions are satisfied:

- (i) $W(0) = C$, and
- (ii) $W(t)W(s) = CW(t+s)$, for all $t, s \geq 0$.

A linear operator $A : D(A) \subseteq X \rightarrow X$ is called the generator of W if $Ax = C^{-1} \lim_{t \rightarrow 0} \frac{W(t)x - Cx}{t}$, where $D(A)$ denotes the maximal domain of A in X . Moreover, if there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|W(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ for all $t \geq 0$, then the semigroup W is said to be of type (M, ω) . Theorem 3.4 of [8] asserts that if W is a C -regularized semigroup generated by A , $x \in D(A)$ and $t \geq 0$. Then $W(t)x = Cx + \int_0^t AW(s)x ds$. In this way, $u(t) := W(t)x$ is a classical solution of the abstract Cauchy problem

$$(\text{ACP}_C) \begin{cases} u'(t) &= Au(t), \\ u(0) &= Cx. \end{cases}$$

Moreover, if $x \in D(A^k)$ for some $k \in \mathbb{N}$, then $u(t) := W(t)x \in C^k([0, \infty), X)$, and $u^{(k)}(t) = A^k u(t)$.

The first problem to face when extending the Hille-Phillips Functional Calculus to generators of C -regularized semigroups is the fact that $f(A)$ might not be a bounded operator if A generates a C -regularized semigroup and $f \in F_\omega$. In order to see this, consider $H_t \in \text{NBV}^\omega$ for all $\omega \in \mathbb{R}$ and $t \geq 0$. It follows that $f_t(z) := e^{tz} = \int_0^\infty e^{sz} dH_t(s) \in F_\omega$, $\omega \in \mathbb{R}$, and $t \geq 0$. If $Af(s) = sf(s)$ on $C_0(\mathbb{R}, \mathbb{C})$, then $f_t(A)$ is an unbounded operator for all $t \geq 0$. The following result shows that, by regularizing in the same way as for C -regularized semigroups, one obtains a C -regularized Hille-Phillips Functional Calculus.

Theorem 2.1 (Regularized HP Functional Calculus). *Let W be a C -regularized semigroup of type (M, ω) on X generated by A . If $f_\alpha \in F_\omega$ is such that $f(z) = \int_0^\infty e^{zt} d\alpha(t)$ ($\text{Re}(z) \leq \omega$), then the map $\Psi : F_\omega \rightarrow \mathcal{L}(X)$ defined by*

$$\Psi(f_\alpha)x := \int_0^\infty W(t)x d\alpha(t)$$

satisfies that $\Psi(f_\alpha)\Psi(f_\beta) = C\Psi(f_{\alpha\beta})$ and $\|\Psi(f_\alpha)\| \leq M\|\alpha\|_\omega$.*

The proof of Theorem 2.1 is carried over by showing that if W is a C -semigroup of type (M, ω) , then the map $x \rightarrow \int_0^\infty W(t)x d\alpha(t) \in \mathcal{L}(X)$ for every $\alpha \in \text{NBV}^\omega$; and then by using integration by parts in order to show that $\langle \Psi(f_\alpha)\Psi(f_\beta)x, \phi \rangle = \langle C\Psi(f_{\alpha*\beta})x, \phi \rangle$ for all $\phi \in X'$ and $x \in X$, for details see [17].

Formally, one thinks of $\Psi(f)$ as the operator $f(A)C$ for $f \in F_\omega$.

Definition 2.2. Let W be a C -regularized semigroup of type (M, ω) on X generated by A . If $\alpha_\lambda(t) := \int_0^t e^{-\lambda s} ds$ for $\text{Re}(\lambda) > \omega$, then $\alpha_\lambda \in \text{NBV}^\omega$ and $f_{\alpha_\lambda}(z) = \int_0^\infty e^{zt} d\alpha_\lambda(t) = \frac{1}{\lambda - z}$ for $\text{Re}(z) \leq \omega$. By Theorem 2.1, one can define the C -resolvent operator $R(\lambda, A)C \in \mathcal{L}(X)$ by $R(\lambda, A)Cx := \Psi(f_{\alpha_\lambda})x$ for all $x \in X$ and $\text{Re}(\lambda) > \omega$; i.e., the Laplace transform of W is the C -resolvent operator.

3. STABILITY AND CONVERGENCE OF REGULARIZED RATIONAL SCHEMES

In 1948, E. Hille showed that if A generates a strongly continuous semigroup T on a Banach space X , then

$$(3.1) \quad T(t)x = \lim_{n \rightarrow \infty} \left(I - \frac{t}{n}A \right)^{-n} x \quad \text{for all } x \in X \text{ and } t \geq 0.$$

In this case $u(t) := T(t)x$ is the (mild) solution of the abstract Cauchy problem $u'(t) = Au(t)$, $u(0) = x$. Therefore, (3.1) provides an approximation to the solution u of (ACP). On the other hand, if r_0 is the subdiagonal Padé approximant given by $r_0(z) = \frac{1}{1-z}$ for $\operatorname{Re}(z) \leq 0$, then (3.1) is equivalent to

$$(3.2) \quad T(t)x = \lim_{n \rightarrow \infty} r_0^n \left(\frac{t}{n}A \right) x \quad \text{for all } x \in X \text{ and } t \geq 0.$$

Since r_0 approximates the exponential function to order $q = 1$, a natural question is to find out if different rational functions r , which approximate the exponential function, provide approximations for the semigroup T and, more importantly, to find error estimates for those approximations. Following the work of R. Hersh and T. Kato [12], P. Brenner and V. Thomée provide an answer to both questions by using the Hille-Phillips functional calculus in [3]. It is shown in [3] that if r is an \mathcal{A} -stable rational approximation to the exponential for which $\|r^n(tA)\|_{\mathcal{L}(X)}$ is uniformly bounded for $n \in \mathbb{N}$ and $t \geq 0$, then

$$(3.3) \quad T(t)x = \lim_{n \rightarrow \infty} r^n \left(\frac{t}{n}A \right) x \quad \text{for all } x \in X \text{ and } t \geq 0,$$

with the appropriate error estimates for smooth initial data; i.e., $x \in D(A^m)$.

In this section we show that the results of Brenner and Thomée can be fully extended in order to approximate C -regularized semigroups W by using the regularized version of the Hille-Phillips functional calculus of Section 2. In this way, we obtain regularized rational approximation schemes for solutions of (ACP $_C$) with error estimates for smooth initial data when A is the generator of a C -regularized semigroup. Furthermore, since the Trotter-Kato Theorem provides space discretizations for approximating strongly continuous semigroups T , we show an extension of it for C -regularized semigroups and we combine the space discretization with the time discretization previously mentioned in order to obtain fully discretized schemes which are convergent to the solution of (ACP $_C$) when A generates a C -semigroup.

Theorem 3.1. *Let W be a C -semigroup of type $(M, 0)$. If r is an \mathcal{A} -stable rational approximation to the exponential of order q , then there exist $K > 0$ such that*

$$(3.4) \quad \|r^n(\rho A)C\|_{\mathcal{L}(X)} \leq KMn^{\frac{1}{2}}, \quad \rho > 0, \quad n \in \mathbb{N}.$$

If, in addition, r is \mathcal{A} -acceptable and satisfies (\star) , then

$$(3.5) \quad \|r^n(\rho A)C\|_{\mathcal{L}(X)} \leq KMn^{\frac{1}{2}(1-\frac{q+1}{p})}, \quad \rho > 0, \quad n \in \mathbb{N}.$$

Moreover, for $1 \leq k \leq q+1$, $k \neq \frac{q-1}{2}$, and $x \in D(A^k)$,

$$(3.6) \quad \left\| r^n \left(\frac{t}{n}A \right) Cx - W(t)x \right\| \leq MKt^{k-\theta_q(k)} \left(\frac{t}{n} \right)^{\theta_q(k)} \|A^k x\|.$$

If $k = \frac{q-1}{2}$ then an extra $\ln(n+1)$ should be considered in (3.6). Furthermore, if in addition, r satisfies the condition (\star) , then θ_q can be replaced by θ_q^ .*

Proof. Let $n \in \mathbb{N}$, $\rho > 0$ and $x \in X$. From Theorem 2.1, one obtains that $\|r^n(\rho A)Cx\| \leq \int_0^\infty \|W(\rho s)x\| dV_{\alpha^{n*}}(s) \leq M\|\alpha^{n*}\|_0\|x\|$. Thus, (3.4) and (3.5) follows from (1.11) and (1.12) respectively. Now, let $x \in D(A^k)$. It follows from Theorem 2.1, equation (1.7), and integrating by parts k -times that

$$\begin{aligned} \left\| r^n \left(\frac{t}{n} A \right) Cx - W(t)x \right\| &= \left\| \int_0^\infty W(s)x d\alpha_{n,t}(s) - \int_0^\infty W(s)x dH_t(s) \right\| \\ &= \left\| \int_0^\infty W(s)x d[\alpha_{n,t} - H_t](s) \right\| = \left\| \int_0^\infty I^{(k-1)}[\alpha_{n,t} - H_t](s) \frac{d^k}{ds^k} [W(s)x] ds \right\| \\ &= \left\| \int_0^\infty I^{(k-1)}[\alpha_{n,t} - H_t](s) W(s) A^k x ds \right\| \leq M \|I^{(k-1)}(\alpha_{n,t} - H_t)\|_{L^1} \|A^k x\|, \end{aligned}$$

and the result of (3.6) follow from (1.13). \square

The estimations obtained in Theorem 3.1 are sharp by considering $A = \frac{d}{dx}$, and $C = I$ on $L^\infty(\mathbb{R})$, see [2]. On the other hand, Theorem 3.1 shows that if r_j is a subdiagonal Padé approximant then $\|r_j^n(\rho A)C\|_{\mathcal{L}(X)}$ is uniformly bounded for $n \in \mathbb{N}$ and $\rho \geq 0$ since, in this case, $\bar{q} = 2j + 1$ and $\bar{p} = 2j + 2$. Therefore, $W(t)x = \lim_{n \rightarrow \infty} r_j^n \left(\frac{t}{n} A \right) Cx$ for all $x \in \overline{D(A)}$, see Corollary 2.4.4 of [17].

The rest of this section is devoted to show an extension of the Trotter-Kato Theorem for C -regularized semigroups, and then combine it with the previously developed time-discretization methods in order to obtain fully discrete approximation schemes convergent to the solution of (ACP_C) .

Theorem 3.2. *Let A, A_n be the generators of C, C_n -regularized semigroups W, W_n of type $(M, 0)$ on X respectively, and suppose that $C_n x \rightarrow Cx$ for all $x \in X$. If $\lim_{n \rightarrow \infty} R(\lambda, A_n)C_n x = R(\lambda, A)Cx$ for all $x \in X$ and some λ with $\operatorname{Re}(\lambda) > 0$, then $\lim_{n \rightarrow \infty} W_n(t)x = W(t)x$, uniformly on $[0, \tau]$, for all $\tau > 0$ and $x \in \overline{\operatorname{Im}(R(\lambda, A)C)}$.*

Proof. By definition, the Laplace transform of W is given by $\widehat{W}(\lambda) = R(\lambda, A)C$. Therefore, it is enough to show that the sequences $\{t \rightarrow W_n(t)x\}_{n \in \mathbb{N}}$ are equicontinuous for each $x \in \operatorname{Im}(R(\lambda, A)C)$ and then apply the Theorem 1.7.5 of [1]. In order to show equicontinuity, an $\frac{\varepsilon}{2}$ -argument will be used. Let $y \in X$ be such that $R(\lambda, A)Cy = x$ and choose n_0 such that $2M\|R(\lambda, A)Cy - R(\lambda, A_n)C_n y\| < \frac{\varepsilon}{2}$ for all $n \geq n_0$. It follows that

$$\begin{aligned} \|W_n(t)x - W_n(s)x\| &= \|W_n(t)R(\lambda, A)Cy - W_n(s)R(\lambda, A)Cy\| \\ &\leq \|W_n(t)R(\lambda, A_n)C_n y - W_n(s)R(\lambda, A_n)C_n y\| + \frac{\varepsilon}{2} \\ &= \left\| \int_0^t W_n(\tau)A_n R(\lambda, A_n)C_n y d\tau - \int_0^s W_n(\tau)A_n R(\lambda, A_n)C_n y d\tau \right\| + \frac{\varepsilon}{2} \\ &\leq M|t - s|\|A_n R(\lambda, A_n)C_n y\| + \frac{\varepsilon}{2} = M|t - s|\|\lambda R(\lambda, A_n)C_n y - C_n y\| + \frac{\varepsilon}{2}. \end{aligned}$$

Since $\sup_{n \in \mathbb{N}} \|\lambda R(\lambda, A_n)C_n y - C_n y\| < \infty$, there exists $\delta > 0$ such that $\|W_n(t)x - W_n(s)x\| \leq \varepsilon$ whenever $|t - s| \leq \delta$ for $n \geq n_0$. Finally, since $t \rightarrow W_n(t)x$ is continuous for $n < n_0$, the equicontinuity follows. \square

Remark 3.3. If $\bigcap_{n \in \mathbb{N}} \rho(A_n) \cap \rho(A)$ is not empty, then $\lim_{n \rightarrow \infty} R(\lambda, A_n) = R(\lambda, A)$ together with $C_n \rightarrow C$ yields that $\lim_{n \rightarrow \infty} W_n(t)x = W(t)x$ uniformly on $[0, \tau]$ for all $\tau > 0$ and $x \in \overline{\operatorname{Im}(R(\lambda, A))} = D(A)$. Therefore, Theorem 3.2 extends Theorem 2.3 of [21] since, for the case where A, A_n generate m -times integrated semigroups of type (M, ω) , $\bigcap_{n \in \mathbb{N}} \rho(A_n) \cap \rho(A) = (\omega, \infty)$, $C_n = R(\lambda, A_n)^m$ for all $n \in \mathbb{N}$, and $C = R(\lambda, A)^m$.

Corollary 3.4. *Let r be an \mathcal{A} -stable rational approximation to the exponential function of order $q \geq 1$. If the conditions of Theorem 3.2 are satisfied then $\lim_{n \rightarrow \infty} r^n \left(\frac{t}{n} A_n \right) C_n x = W(t)x$, for all $t \geq 0$ and $x \in \text{Im}(R(\lambda, A)C)$.*

Proof. Let $x \in \text{Im}(R(\lambda, A)C)$. Since A_m generates a C_m -regularized semigroup of type $(M, 0)$, it follows from Theorem 3.1 with $k = 1$ that for all $t \geq 0$ and $m \in \mathbb{N}$, $\lim_{n \rightarrow \infty} r^n \left(\frac{t}{n} A_m \right) C_m x = W_m(t)x$. On the other hand, from Theorem 3.2 one obtains that $\left\| r^n \left(\frac{t}{n} A_m \right) C_m x - r^n \left(\frac{t}{n} A \right) C x \right\| \leq \int_0^\infty \|W_m(s)x - W(s)x\| dV_{\alpha_n, t}(s) \rightarrow 0$, as $m \rightarrow \infty$ for all $t \geq 0$, $n \in \mathbb{N}$. The Corollary follows by a diagonal argument. \square

It is customary in numerical analysis to approximate operators by considering a sequence of Banach spaces (not always finite dimensional) by means of the finite element or finite difference method. The last part of this section is devoted to show a first step in order to use Theorem 3.2 in such cases.

Let X_n be Banach spaces with norms $\|\cdot\|_n$. From now on, the following conditions are assumed: For every $n \in \mathbb{N}$ there exist bounded linear operators $P_n : X \rightarrow X_n$, and $E_n : X_n \rightarrow X$ such that

- (1) $\|P_n\| \leq N$, $\|E_n\| \leq N'$, where N and N' are independent of n ,
- (2) $\|E_n P_n x - x\| \rightarrow 0$ as $n \rightarrow \infty$ for every $x \in X$, and
- (3) $P_n E_n = I_n$, where I_n is the identity operator on X_n .

Furthermore, a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X_n converges to $x \in X$ if $\|P_n x - x_n\|_n \rightarrow 0$ as $n \rightarrow \infty$. This type of convergence will be denoted by $x_n \rightrightarrows x$. Now, a sequence of linear operators $\{A_n\}_{n \in \mathbb{N}}$ converges to an operator A if $D(A) = \{x : P_n x \in D(A_n)\}$, and $A_n P_n x$ converges, and $Ax = \lim_{n \rightarrow \infty} A_n P_n x$ for every $x \in D(A)$. This type of convergence will be denoted by $A_n \rightrightarrows A$ as $n \rightarrow \infty$.

Theorem 3.5. *Let W be a C -regularized semigroup of type $(M, 0)$ generated by A , and let $A_n : X_n \rightarrow X_n$ be the generator of a C_n -regularized semigroup W_n on X_n of type $(M', 0)$ such that $R(\lambda, A_n)C_n x \rightrightarrows R(\lambda, A)C x$ for all $x \in X$ and $C_n \rightrightarrows C$. If r is an \mathcal{A} -stable rational approximation to the exponential of order q , then $r^n \left(\frac{t}{n} A_n \right) C_n x \rightrightarrows W(t)x$, for all $t \geq 0$ and $x \in \text{Im}(R(\lambda, A)C)$.*

Proof. Let $\widetilde{W}_n(t) : X \rightarrow X$ be defined by $\widetilde{W}_n(t) := E_n W_n(t) P_n$ for each $t \geq 0$. It follows that \widetilde{W}_n is a \widetilde{C}_n -regularized semigroup of type $(NN'M', 0)$, where $\widetilde{C}_n := E_n C_n P_n$. Moreover, $\widetilde{A}_n := E_n A_n P_n$ generates \widetilde{W}_n . On the other hand, \widetilde{A}_n and A generate regularized semigroups of type $(M'', 0)$, where $M'' = \max\{M, NN'M'\}$. Now, let $x \in X$ and $\lambda > 0$. It follows that $R(\lambda, \widetilde{A}_n)\widetilde{C}_n x = \int_0^\infty e^{-\lambda t} \widetilde{W}_n(t) x dt = \int_0^\infty e^{-\lambda t} E_n W_n(t) P_n x dt = E_n \int_0^\infty e^{-\lambda t} W_n(t) P_n x dt = E_n R(\lambda, A_n) C_n P_n x$, and since $R(\lambda, A_n) C_n \rightrightarrows R(\lambda, A) C$ one obtains that $R(\lambda, \widetilde{A}_n)\widetilde{C}_n \rightarrow R(\lambda, A) C$, as $n \rightarrow \infty$. Similarly, $\widetilde{C}_n \rightarrow C$, as $n \rightarrow \infty$. Therefore, if $x \in \overline{\text{Im}(R(\lambda, A)C)}$ one obtains that $\widetilde{W}_n(t)x \rightarrow W(t)x$, as $n \rightarrow \infty$, uniformly on $[0, t]$ for all $t > 0$ by Theorem 3.2. Furthermore, from Corollary 3.4, it follows that $\lim_{n \rightarrow \infty} r^n \left(\frac{t}{n} \widetilde{A}_n \right) \widetilde{C}_n x = W(t)x$ for any \mathcal{A} -stable rational approximation to the exponential of order q . On the other hand, if $\alpha \in \text{NBV}_{\text{loc}}$ and $r(z) = \int_0^\infty e^{zs} d\alpha(s)$ for $\text{Re}(z) \leq 0$, then $r^n \left(\frac{t}{n} \widetilde{A}_n \right) \widetilde{C}_n x = \int_0^\infty \widetilde{W}_n(s) x d\alpha_{n, t}(s) = E_n r^n \left(\frac{t}{n} A_n \right) C_n P_n x$ for all $x \in X$. Thus, $r^n \left(\frac{t}{n} A_n \right) C_n P_n x = P_n r^n \left(\frac{t}{n} \widetilde{A}_n \right) \widetilde{C}_n x$ for all $x \in X$. In this way, $\left\| r^n \left(\frac{t}{n} A_n \right) C_n P_n x - P_n W(t)x \right\|_n =$

$\left\| P_n r^n \left(\frac{t}{n} \widetilde{A}_n \right) \widetilde{C}_n x - P_n W(t)x \right\|_n \leq N \left\| r^n \left(\frac{t}{n} \widetilde{A}_n \right) \widetilde{C}_n x - W(t)x \right\| \rightarrow 0$, as $n \rightarrow \infty$ for any $x \in \text{Im}(R(\lambda, A)C)$. \square

4. APPLICATIONS TO INTEGRATED SEMIGROUPS

In this section we apply Theorem 3.1 to generators of integrated semigroups. We follow [1] for the definitions and basic results concerning this theory. Let $m \in \mathbb{N}_0$. A linear operator A generates an m -times integrated semigroup if there exists $\omega \geq 0$ and a strongly continuous function $S : \mathbb{R}_0^+ \rightarrow \mathcal{L}(X)$ such that $(\omega, \infty) \subset \rho(A)$ and $R(\lambda, A) = \lambda^m \int_0^\infty e^{-\lambda t} S(t) dt$, ($\lambda > \omega$). In this case S is called a m -times integrated semigroup generated by A . If there exist $M, \omega \geq 0$ such that $\| \int_0^t S(s) ds \| \leq M e^{\omega t}$, then S is of type (M, ω) . An important example is provided by M. M. H. Pang in [22] where it is shown that the Schrödinger operator $\Delta - V$ generates an $N+2$ -times integrated semigroup on $L^p(\mathbb{R}^N)$ for $1 \leq p \leq \infty$ if the potential V satisfies that $V_+ \in K_{\text{loc}}^N$ and $V_- \in K^N$, where K^N is the Kato class, see [22] for more details. Theorem 18.3 of [8] asserts that if A generates an m -times integrated semigroup S of type (M, ω) , then $W(t) = \frac{d^m}{dt^m} S(t) R(\mu, A)^m$ is a $R(\mu, A)^m$ -semigroup of type (M, ω) on X for every $\mu \in \rho(A)$. Furthermore, $S(t) R(\mu, A)^m x = I^m[W](t)x$. Formally, the definition suggests that an m -times integrated semigroup is of the form $S(t) = I^m(T)(t)$, where T is a semigroup on X for which $T(t)$ might not be in $\mathcal{L}(X)$. Since one can approximate a strongly continuous semigroup or bi-continuous semigroup T (with error estimates) by \mathcal{A} -stable rational functions which approximate the exponential (see [3, 16]), it is natural to ask if there is an integrated version of such schemes for integrated semigroups. The following result gives an affirmative answer to that by providing error estimates for integrated schemes of m -times integrated semigroups for smooth initial data.

Theorem 4.1. *Let A be the generator of an m -times integrated semigroup S of type $(M, 0)$. If r is an \mathcal{A} -stable rational approximation to the exponential of order q , then there exists $K > 0$ such that*

$$(4.1) \quad \left\| I^m \left[r^n \left(\frac{\cdot}{n} A \right) \right] (t)y - S(t)y \right\| \leq MK \frac{t^{k+m-\theta_q(k)}}{k+m} \left(\frac{t}{n} \right)^{\theta_q(k)} \|A^{k+m}y\|,$$

for $1 \leq k \leq q+1$, $k \neq \frac{q-1}{2}$, and $y \in D(A^{k+m})$. If $k = \frac{q-1}{2}$ then one obtains an extra $\ln(n+1)$ in (4.1). Moreover, if r satisfies (\star) then θ_q can be replaced by θ_q^* .

Proof. Let $\mu = 0$ (by translating A if necessary), and let $y \in D(A^{k+m})$. Then there exist $x \in X$ such that $A^m y = x$. The result follows by integrating (3.6) applied to $R(0, A)^m x$ and by using that $S(t)y = S(t)R(0, A)^m x = I^m[W](t)x$. \square

Let $x \in X$, $y \in X$, and consider the second order Cauchy problem

$$P^2(x, y) \begin{cases} u''(t) &= Au(t) & (t \geq 0) \\ u(0) &= x, \\ u'(0) &= y. \end{cases}$$

where A is a closed operator on X and generating a cosine function. By transforming the problem $P^2(x, y)$ into a first order system, one obtains that $P^2(x, y)$ is equivalent to the Cauchy problem given by $v'(t) = \mathcal{A}v(t)$, $v(0) = (x, y)$ on the Banach

space $X \times X$ with norm $\|(x, y)\|_{X \times X} = \|x\|_X + \|y\|_X$; where $D(\mathcal{A}) := D(A) \times X$, and

$$\mathcal{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Moreover, if $\lambda^2 \in \rho(A)$ then $\lambda \in \rho(\mathcal{A})$ and

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} \lambda R(\lambda^2, A) & R(\lambda^2, A) \\ AR(\lambda^2, A) & \lambda R(\lambda^2, A) \end{pmatrix}.$$

Theorem 3.14.7 of [1] asserts that an operator A generates a cosine function on X if and only if \mathcal{A} generates a once integrated semigroup S on $X \times X$. Thus, \mathcal{A} generates a $R(\mu, \mathcal{A})$ -semigroup of type (M, ω) for any $\mu \in \rho(\mathcal{A})$. Now, by translating if necessary, one can assume that $\omega = 0$. Therefore, by Theorem 3.1, \mathcal{A} -stable rational functions provide approximation methods for the solutions of the Cauchy problem $v'(t) = \mathcal{A}v(t)$; $v(0) = R(\mu, \mathcal{A})(z, w)$, for any $(z, w) \in D(\mathcal{A})$. In particular, if $r(z) = \frac{1}{1-z}$ for $\operatorname{Re}(z) \leq 0$, then r is an approximation of order $q = 1$ to the exponential. On the other hand, if $x \in D(A^2)$ and $y \in D(A)$, then there exist $(z, w) \in D(\mathcal{A})$ such that $R(\mu, \mathcal{A})(z, w) = (x, y)$, since $\operatorname{Im}(R(\mu, \mathcal{A})) = D(\mathcal{A})$. Therefore, by considering $k = 1$ in Theorem 3.1, there exist $K > 0$ such that the solution $u(\cdot, x, y)$ of the second order problem $P^2(x, y)$ can be approximated by $r^n(\frac{t}{n}, \mathcal{A}) = (\frac{n}{t})^n V^n(\frac{n}{t}, \mathcal{A})$ in the following way:

$$(4.2) \quad \left\| \left(\frac{n}{t} \right)^n V^n \left(\frac{n}{t}, \mathcal{A} \right) (x, y) - u(t, x, y) \right\| \leq MK \frac{t}{\sqrt{n}} \|\mathcal{A}(z, w)\|,$$

where

$$\begin{aligned} V^n(\lambda, \mathcal{A})(x, y) &:= \sum_{j=1}^m \binom{2m-1}{2j-1} \lambda^{2j-1} A^{m-j} R(\lambda^2, A)^n x \\ &\quad + \sum_{j=0}^{m-1} \binom{2m-1}{2j} \lambda^{2j} A^{m-j-1} R(\lambda^2, A)^n y, \end{aligned}$$

for n odd and $m := \frac{n+1}{2}$. If n is even and $m := \frac{n}{2}$ then

$$\begin{aligned} V^n(\lambda, \mathcal{A})(x, y) &= \sum_{j=0}^m \binom{2m}{2j} \lambda^{2j} A^{m-j} R(\lambda^2, A)^n x \\ &\quad + \sum_{j=1}^m \binom{2m}{2j-1} \lambda^{2j-1} A^{m-j} R(\lambda^2, A)^n y. \end{aligned}$$

The formula (4.2) corresponds to the Backward-Euler scheme for the second order problem $P^2(x, y)$ since $r(z) = \frac{1}{1-z}$. Similarly, if $r(z) = \frac{2+z}{2-z}$ for $\operatorname{Re}(z) \leq 0$, then Theorem 3.1 (with $q = 2$ and $k = 3$) provides an explicit form for the Crank-Nicolson approximation scheme for the solution of the second order problem $P^2(x, y)$ by

$$\left\| U^n \left(\frac{n}{t}, \mathcal{A} \right) (x, y) - u(t, x, y) \right\| \leq MK \frac{t}{n^2} \|\mathcal{A}^3(z, w)\|,$$

where

$$U^n \left(\frac{n}{t}, \mathcal{A} \right) = \sum_{k=0}^n \kappa_k^n (-1)^{n-k} \left(\frac{4n}{t} \right)^k V^k \left(\frac{2n}{t}, \mathcal{A} \right),$$

and $\kappa_{\beta}^{\alpha} := \frac{\alpha!}{(\alpha-\beta)! \beta!}$.

Notice that in the case of generators of once integrated semigroups, it is known that there exist a Banach space V such that $D(A) \hookrightarrow V \hookrightarrow X$ and such that the part \mathcal{A} in $V \times X$ generates a strongly continuous semigroup, see Theorem 3.14.11 of [1]. However, even though the same estimations are obtained when applying the results of P. Brenner and V. Thomée for strongly continuous semigroups, the constant K will also depend on the phase space V . Theorem 3.1 shows that, by regularizing r , the constants K depend only on the \mathcal{A} -stable rational function.

Finally, Figure 1 illustrates the backward-Euler scheme for the one dimensional wave equation by considering $A = \frac{d^2}{ds^2}$, $x = \sin$, and $y = 0$ on the Banach space of bounded continuous functions defined on \mathbb{R} with the uniform norm. In [13], M. Hieber shows that \mathcal{A} generates a once integrated semigroup on $C_b(\mathbb{R})$. The solution $u(t, s) = \sin(s) \cos(t)$ is solid-colored and the approximation is mesh-colored.

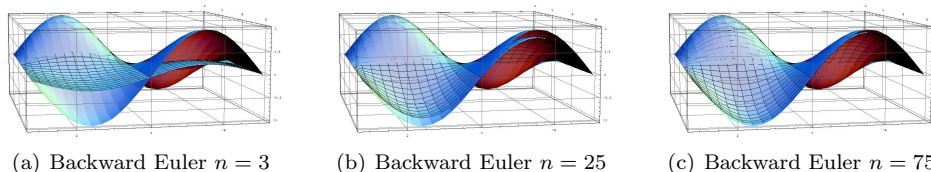


FIGURE 1. Backward Euler scheme (4.2) for the wave equation on $C_b(\mathbb{R})$.

ACKNOWLEDGEMENTS

The author acknowledges the support provided by the Louisiana Education Quality Support Fund LEQSF(2005-2007)-ENH-TR-21 for visiting the University of Parma – Italy, the Universities of Karlsruhe and Tübingen – Germany, and the Delft University of Technology – The Netherlands. He also would like to thank Markus Haase and Frank Neubrander for their comments. The helpful comments and suggestions of the anonymous referee are gratefully acknowledged.

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DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70803
 E-mail address: pjara@math.lsu.edu